Mathematical Modelling on RLCG Transmission Lines*

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Abstract. A new model on RLCG transmissions lines is presented in the paper. The model suits to be taken to directly simulating a circuit system in time-domain. Mathematically, a circuit system with distributed elements may be described by a special kind of nonlinear integral-differential-algebraic equations with multiple constant delays.

Keywords: RLCG transmission lines, modelling, nonlinear circuits, integral-differential-algebraic equations with multiple delays, simulation in time-domain.

1 Introduction

It is known that the conductors of a circuit system should be regarded as transmission lines for theoretical analysis and practical design in the recent high-speed integrated circuit technology [1]. At relatively higher signal-speeds, transmission line models based on quasi-transverse electro-magnetic mode (TEM) assumptions are severely useful for circuit simulation [2]. The TEM approximation represents the ideal case. Often, from the system design point of view the solution to Maxwell’s equations may be given by the so-called quasi-TEM modes, and it can be characterized by distributed parameters R, L, C, and G [3].

The simulation task is to compute the transient response of a circuit system consisting of nonlinear devices interconnected by transmission lines. In the short

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paper we establish a new model on RLCG transmission lines in time-domain. A circuit system with distributed elements may be described by nonlinear integral-differential-algebraic equations (IDAEs) with multiple constant delays such that the general-purpose circuit simulators can be then used to solve the new equations in theory.

2 Distributed models of RLCG transmission lines

In general, a transmission line is presented by Telegrapher’s equations. For an RLCG transmission line system shown in Fig. 1, at time $t$ ($0 \leq t \leq T_e$) let $v(x, t)$ and $i(x, t)$ respectively be voltage and current at point $x$ ($0 \leq x \leq d$). The basic equations are

$$\frac{\partial v(x, t)}{\partial x} + L \frac{\partial i(x, t)}{\partial t} = -R i(x, t),$$  
$$\frac{\partial i(x, t)}{\partial x} + C \frac{\partial v(x, t)}{\partial t} = -G v(x, t),$$

where $R$ is resistance, $L$ is inductance, $C$ is capacitance, and $G$ is conductance for unit length. The constants $R$, $L$, $C$, and $G$ are distributed parameters for RLCG transmission line. If $R = 0$ and $G = 0$, the transmission line is lossless, see [4].

![Fig. 1. An RLCG transmission line.](image)

Let $LC = 1/\nu^2$ where $\nu$ is velocity of signal propagation, $\tau = \frac{d}{\nu}$ which is the delay of a signal going from $x = 0$ to $x = d$, and $z_0 = \sqrt{L/C}$ which is the characteristic impedance. We now write (1) in matrix form as follows

$$\frac{\partial U(x, t)}{\partial x} + \bar{A} \frac{\partial U(x, t)}{\partial t} = \bar{B} U(x, t),$$

(2)
where $U(x, t) = [v(x, t), i(x, t)]^t$, and

$$\tilde{A} = \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & -R \\ -G & 0 \end{bmatrix}. $$

Obviously, we have

$$T\tilde{A}T^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix},$$

where $\lambda = \sqrt{LC}$ and

$$T = \frac{1}{2} \begin{bmatrix} 1 & z_0 \\ 1 & -z_0 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 \\ -z_0 & 1 \end{bmatrix}. \quad (3)$$

We let $\tilde{U}(x, t) = TU(x, t)$ where $\tilde{U}(x, t) = [\tilde{U}_v(x, t), \tilde{U}_i(x, t)]^t$, from (2) we have

$$\frac{\partial \tilde{U}(x, t)}{\partial x} + \tilde{D} \frac{\partial \tilde{U}(x, t)}{\partial t} = \tilde{E} \tilde{U}(x, t),$$

where $\tilde{D} = T\tilde{A}T^{-1}$ and $\tilde{E} = T\tilde{B}T^{-1}$. Namely,

$$\frac{\partial \tilde{U}_v(x, t)}{\partial x} + \lambda \frac{\partial \tilde{U}_v(x, t)}{\partial t} = (\tilde{E} \tilde{U}(x, t))_1, \quad (4)$$

$$\frac{\partial \tilde{U}_i(x, t)}{\partial x} - \lambda \frac{\partial \tilde{U}_i(x, t)}{\partial t} = (\tilde{E} \tilde{U}(x, t))_2,$$

where $(\tilde{E} \tilde{U}(x, t))_1$ and $(\tilde{E} \tilde{U}(x, t))_2$ are the two elements of $\tilde{E} \tilde{U}(x, t)$. The above partial differential equations (PDEs) can be further expressed as a form of integral equations with constant delay by the method of characteristics (MC), see [5].

First, we construct two characteristic lines as follows

$$l_+ : \frac{dt}{dx} = \lambda,$$

$$l_- : \frac{dt}{dx} = -\lambda.$$ 

In other words, the lines $l_+$ and $l_-$ are defined by $t - \lambda x = c$ and $t + \lambda x = c$, where $c$ is some constant. For any point $(x, t)$, we integrate the first equation and
the second equation of (4) along the lines $l_+$ and $l_-$ respectively from $(0, t - \lambda x)$ and $(0, t + \lambda x)$. Thus,

\[
\tilde{U}_v(x, t) = \tilde{U}_v(0, t - \lambda x) + \int_{l_+}^{x} \left( \tilde{E}\tilde{U} (l, t - \lambda(x - l)) \right) dl,
\]

\[
\tilde{U}_i(x, t) = \tilde{U}_i(0, t + \lambda x) + \int_{l_-}^{x} \left( \tilde{E}\tilde{U} (l, t + \lambda(x - l)) \right) dl,
\]

where $f$ and $g$ are two continuously differentiably initial functions.

Based on (5), by $\tilde{E} = T\tilde{B}T^{-1}$ and $\tilde{U} = TU$ we have

\[
TU(x, t) = \left[ \begin{array}{c} f(t - \lambda x) \\ g(t + \lambda x) \end{array} \right] + \int_0^{x} \left[ \begin{array}{c} (T\tilde{B}U(l, t - \lambda(x - l)))_1 \\ (T\tilde{B}U(l, t + \lambda(x - l)))_2 \end{array} \right] dl.
\]

Since

\[
T\tilde{B}U = \frac{1}{2} \begin{bmatrix} -z_0G & -R \\ z_0G & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} z_0Gv + Ri \\ -z_0Gv + Ri \end{bmatrix},
\]

from (6) we have

\[
f(t - \lambda x) = \frac{1}{2} \left[ v(x, t) + z_0i(x, t) \right] + \frac{1}{2} \left[ z_0G \int_0^{x} v(l, t - \lambda(x - l)) dl + R \int_0^{x} i(l, t - \lambda(x - l)) dl \right],
\]

\[
g(t + \lambda x) = \frac{1}{2} \left[ v(x, t) - z_0i(x, t) \right] - \frac{1}{2} \left[ z_0G \int_0^{x} v(l, t + \lambda(x - l)) dl - R \int_0^{x} i(l, t + \lambda(x - l)) dl \right].
\]
Let $x = d$ in the second equation of (8), it follows

$$
g(t + \lambda d) = \frac{1}{2} [v(d, t) - z_0 i(d, t)] - \frac{1}{2} \left[ z_0 G \int_0^d v(l, t + \lambda(d - l)) dl - R \int_0^d i(l, t + \lambda(d - l)) dl \right].
$$

Let $x = 0$ in the first equation of (8) and set $t = t - \lambda d$ in the above expression, by $\lambda d = \tau$ we can arrive at

$$
f(t) = \frac{1}{2} W_B(t),
$$

$$
g(t) = \frac{1}{2} W_A(t - \tau) - \frac{1}{2} \left[ z_0 G \int_0^d v(l, t - \lambda l) dl - R \int_0^d i(l, t - \lambda l) dl \right],
$$

where the new functions $W_A$ and $W_B$ are defined as

$$
W_A(t) = v(d, t) - z_0 i(d, t), \quad W_B(t) = v(0, t) + z_0 i(0, t).
$$

Now, by (6), (7), and the form of $T^{-1}$ in (5) we know

$$
v(x, t) = f(t - \lambda x) + g(t + \lambda x)
$$

$$
- \frac{z_0 G}{2} \int_0^x \left[ v(l, t - \lambda(x - l)) - v(l, t + \lambda(x - l)) \right] dl
$$

$$
- \frac{R}{2} \int_0^x \left[ i(l, t - \lambda(x - l)) + i(l, t + \lambda(x - l)) \right] dl,
$$

$$
i(x, t) = \frac{1}{z_0} \left[ f(t - \lambda x) - g(t + \lambda x) \right]
$$

$$
- \frac{G}{2} \int_0^x \left[ v(l, t - \lambda(x - l)) + v(l, t + \lambda(x - l)) \right] dl
$$

$$
- \frac{R}{2z_0} \int_0^x \left[ i(l, t - \lambda(x - l)) - i(l, t + \lambda(x - l)) \right] dl.
$$
Then, by (9) it further deduces

\[ v(x, t) = \frac{1}{2} W_A(t - \tau + \lambda x) + \frac{1}{2} W_B(t - \lambda x) \]

\[ - \frac{z_0 G}{2} \left[ \int_0^x v(l, t - \lambda(x - l)) dl + \int_x^d v(l, t + \lambda(x - l)) dl \right] \]

\[ - \frac{R}{2} \left[ \int_0^x i(l, t - \lambda(x - l)) dl - \int_x^d i(l, t + \lambda(x - l)) dl \right], \]

\[ i(x, t) = -\frac{1}{2z_0} W_A(t - \tau + \lambda x) + \frac{1}{2z_0} W_B(t - \lambda x) \]

\[ - \frac{G}{2} \left[ \int_0^x v(l, t - \lambda(x - l)) dl - \int_x^d v(l, t + \lambda(x - l)) dl \right] \]

\[ - \frac{R}{2z_0} \left[ \int_0^x i(l, t - \lambda(x - l)) dl + \int_x^d i(l, t + \lambda(x - l)) dl \right], \]

0 ≤ x ≤ d, 0 ≤ t ≤ T_e.

Based on the second equation of (11), by \( \lambda d = \tau \) we also have

\[ i(0, t) = -\frac{1}{2z_0} W_A(t - \tau) + \frac{1}{2z_0} W_B(t) \]

\[ + \frac{G}{2} \int_0^d v(l, t - \lambda l) dl - \frac{R}{2z_0} \int_0^d i(l, t - \lambda l) dl, \]

\[ i(d, t) = -\frac{1}{2z_0} W_A(t) + \frac{1}{2z_0} W_B(t - \tau) \]

\[ - \frac{G}{2} \int_0^d v(l, t - \tau + \lambda l) dl - \frac{R}{2z_0} \int_0^d i(l, t - \tau + \lambda l) dl. \]

By use of the expressions on \( W_A(t) \) and \( W_B(t) \) in (10), for \( t \in [0, T_e] \) we obtain
the currents at the near and far ends of the line as

$$i(0, t) = \frac{1}{z_0} v(0, t) - \frac{1}{z_0} W_A(t - \tau)$$

$$+ G \int_0^d v(l, t - \lambda l) dl - \frac{R}{z_0} \int_0^d i(l, t - \lambda l) dl,$$

$$i(d, t) = -\frac{1}{z_0} v(d, t) + \frac{1}{z_0} W_B(t - \tau)$$

$$- G \int_0^d v(l, t - \tau + \lambda l) dl - \frac{R}{z_0} \int_0^d i(l, t - \tau + \lambda l) dl.$$  \hfill (12)

This is a basic characteristic for RLCG transmission lines.

From (12), for $t \in [0, T_e]$ we also have the voltages at the near and far ends of the line as

$$v(0, t) = z_0 i(0, t) + W_A(t - \tau)$$

$$- z_0 G \int_0^d v(l, t - \lambda l) dl + R \int_0^d i(l, t - \lambda l) dl,$$

$$v(d, t) = -z_0 i(d, t) + W_B(t - \tau)$$

$$- z_0 G \int_0^d v(l, t - \tau + \lambda l) dl - R \int_0^d i(l, t - \tau + \lambda l) dl.$$  \hfill (13)

To combine (10) and (13), we further arrive at

$$W_A(t) = 2v(d, t) - W_B(t - \tau)$$

$$+ z_0 G \int_0^d v(l, t - \tau + \lambda l) dl + R \int_0^d i(l, t - \tau + \lambda l) dl,$$

$$W_B(t) = 2v(0, t) - W_A(t - \tau)$$

$$+ z_0 G \int_0^d v(l, t - \lambda l) dl - R \int_0^d i(l, t - \lambda l) dl.$$
3 A general form of circuit equations with distributed elements

We first see two simple circuits with $RLCG$ transmission lines. For the basic circuit with distributed elements shown in Fig. 2, its circuit equations are

\[
c_1 \frac{dv_1(t)}{dt} = g_1(e - v_1(t)) - \frac{1}{z_0} v_1(t) + \frac{1}{z_0} W_A(t - \tau)
- G \int_0^d v(l, t - \lambda l) dl + \frac{R}{z_0} \int_0^d i(l, t - \lambda l) dl,
\]

\[
c_2 \frac{dv_2(t)}{dt} = -g_2(v_2(t)) - \frac{1}{z_0} v_2(t) + \frac{1}{z_0} W_B(t - \tau)
- G \int_0^d v(l, t - \lambda l) dl - \frac{R}{z_0} \int_0^d i(l, t - \lambda l) dl,
\]

\[
W_A(t) = 2v_2(t) - W_B(t - \tau)
- \frac{z_0 G}{2} \int_0^d v(l, t - \lambda l) dl + \frac{R}{z_0} \int_0^d i(l, t - \lambda l) dl,
\]

\[
W_B(t) = 2v_1(t) - W_A(t - \tau)
+ \frac{z_0 G}{2} \int_0^d v(l, t - \lambda l) dl - \frac{R}{z_0} \int_0^d i(l, t - \lambda l) dl,
\]

\[
v(x,t) = \frac{1}{2} W_A(t - \tau + \lambda x) + \frac{1}{2} W_B(t - \lambda x)
- \frac{z_0 G}{2} \left[ \int_0^x v(l, t - \lambda (x - l)) dl + \int_0^x v(l, t + \lambda (x - l)) dl \right]
- \frac{R}{2} \left[ \int_0^x i(l, t - \lambda (x - l)) dl - \int_0^x i(l, t + \lambda (x - l)) dl \right],
\]

\[
i(x,t) = -\frac{1}{2z_0} W_A(t - \tau + \lambda x) + \frac{1}{2z_0} W_B(t - \lambda x)
- \frac{G}{2} \left[ \int_0^x v(l, t - \lambda (x - l)) dl - \int_0^x v(l, t + \lambda (x - l)) dl \right]
- \frac{R}{2z_0} \left[ \int_0^x i(l, t - \lambda (x - l)) dl + \int_0^x i(l, t + \lambda (x - l)) dl \right],
\]
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\( v(0, t) = v_1(t), \quad v(d, t) = v_2(t), \quad 0 \leq x \leq d, \quad 0 \leq t \leq T_e, \)

where \( g_1 \) and \( g_2 \) are nonlinear functions.

\[\begin{align*}
\frac{dv_1(t)}{dt} &= -\left( \frac{1}{R_1} + \frac{1}{\tau_1} \right) v_1(t) + \frac{1}{\tau_1} W_{A1}(t - \tau_1)
- G_{W1} \int_0^{d_1} v_{W1}(l, t - \lambda_1 l) dl + \frac{R_{W1}}{\tau_1} \int_0^{d_1} i_{W1}(l, t - \lambda_1 l) dl + \frac{e(t)}{R_0}, \\
\frac{dv_2(t)}{dt} &= -\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) v_2(t) + \frac{1}{\tau_2} W_{B1}(t - \tau_1) + \frac{1}{\tau_2} W_{A2}(t - \tau_2)
- G_{W1} \int_0^{d_1} v_{W1}(l, t - \tau_1 + \lambda_1 l) dl - \frac{R_{W1}}{\tau_1} \int_0^{d_1} i_{W1}(l, t - \tau_1 + \lambda_1 l) dl \\
&- G_{W2} \int_0^{d_2} v_{W2}(l, t - \lambda_2 l) dl + \frac{R_{W2}}{\tau_2} \int_0^{d_2} i_{W2}(l, t - \lambda_2 l) dl,
\end{align*}\]

\[\begin{align*}
\frac{dv_3(t)}{dt} &= -\left( \frac{1}{R_f} + \frac{1}{\tau_2} \right) v_3(t) + \frac{1}{\tau_2} W_{B2}(t - \tau_2)
- G_{W2} \int_0^{d_2} v_{W2}(l, t - \tau_2 + \lambda_2 l) dl - \frac{R_{W2}}{\tau_2} \int_0^{d_2} i_{W2}(l, t - \tau_2 + \lambda_2 l) dl,
\end{align*}\]

\( W_{A1}(t) = 2v_2(t) - W_{B1}(t - \tau_1) \)

\( + \tau_0 G_{W1} \int_0^{d_1} v_{W1}(l, t - \tau_1 + \lambda_1 l) dl + R_{W1} \int_0^{d_1} i_{W1}(l, t - \tau_1 + \lambda_1 l) dl, \)

Fig. 2. A circuit with RLCG transmission lines.

Another distributed circuit is shown in Fig. 3. Its circuit equations are described by IDAEs with multiple constant delays,

\[\begin{align*}
\begin{aligned}
c_1 \frac{dv_1(t)}{dt} &= -\left( \frac{1}{R_1} + \frac{1}{\tau_1} \right) v_1(t) + \frac{1}{\tau_1} W_{A1}(t - \tau_1) \\
&- G_{W1} \int_0^{d_1} v_{W1}(l, t - \lambda_1 l) dl + \frac{R_{W1}}{\tau_1} \int_0^{d_1} i_{W1}(l, t - \lambda_1 l) dl + \frac{e(t)}{R_0}, \\
\frac{dv_2(t)}{dt} &= -\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) v_2(t) + \frac{1}{\tau_2} W_{B1}(t - \tau_1) + \frac{1}{\tau_2} W_{A2}(t - \tau_2) \\
&- G_{W1} \int_0^{d_1} v_{W1}(l, t - \tau_1 + \lambda_1 l) dl - \frac{R_{W1}}{\tau_1} \int_0^{d_1} i_{W1}(l, t - \tau_1 + \lambda_1 l) dl \\
&- G_{W2} \int_0^{d_2} v_{W2}(l, t - \lambda_2 l) dl + \frac{R_{W2}}{\tau_2} \int_0^{d_2} i_{W2}(l, t - \lambda_2 l) dl,
\end{aligned}
\end{align*}\]
\[ W_{B1}(t) = 2v_1(t) - W_{A1}(t - \tau_1) \]
\[ + z_{01}G_{W1} \int_{0}^{d_1} v_W(l, t - \lambda l) dl - R_{W1} \int_{0}^{d_1} i_W(l, t - \lambda l) dl, \]
\[ W_{A2}(t) = 2v_3(t) - W_{B2}(t - \tau_2) \]
\[ + z_{02}G_{W2} \int_{0}^{d_2} v_W(l, t + \tau_2 + \lambda l) dl + R_{W2} \int_{0}^{d_2} i_W(l, t + \tau_2 + \lambda l) dl, \]
\[ W_{B2}(t) = 2v_2(t) - W_{A2}(t - \tau_2) \]
\[ + z_{02}G_{W2} \int_{0}^{d_2} v_W(l, t - \lambda l) dl - R_{W2} \int_{0}^{d_2} i_W(l, t - \lambda l) dl, \]
\[ v_W(x, t) = \frac{1}{2} W_{A1}(t - \tau_1 + \lambda_1 x) + \frac{1}{2} W_{B1}(t - \lambda_1 x) \]
\[ - \frac{z_{01}G_{W1}}{2} \left[ \int_{0}^{x} v_W(l, t - \lambda_1(x - l)) dl + \int_{x}^{d_1} v_W(l, t + \lambda_1(x - l)) dl \right] \]
\[ - \frac{R_{W1}}{2} \left[ \int_{0}^{x} i_W(l, t - \lambda_1(x - l)) dl - \int_{x}^{d_1} i_W(l, t + \lambda_1(x - l)) dl \right], \]
\[ i_W(x, t) = - \frac{1}{2z_{01}} W_{A1}(t - \tau_1 + \lambda_1 x) + \frac{1}{2z_{01}} W_{B1}(t - \lambda_1 x) \]
\[ - \frac{G_{W1}}{2} \left[ \int_{0}^{x} v_W(l, t - \lambda_1(x - l)) dl - \int_{x}^{d_1} v_W(l, t + \lambda_1(x - l)) dl \right] \]
\[ - \frac{R_{W1}}{2z_{01}} \left[ \int_{0}^{x} i_W(l, t - \lambda_1(x - l)) dl + \int_{x}^{d_1} i_W(l, t + \lambda_1(x - l)) dl \right], \]
\[ v_W(y, t) = \frac{1}{2} W_{A2}(t - \tau_2 + \lambda_2 y) + \frac{1}{2} W_{B2}(t - \lambda_2 y) \]
\[ - \frac{z_{02}G_{W2}}{2} \left[ \int_{0}^{y} v_W(l, t - \lambda_2(y - l)) dl + \int_{y}^{d_2} v_W(l, t + \lambda_2(y - l)) dl \right] \]
\[ - \frac{R_{W2}}{2} \left[ \int_{0}^{y} i_W(l, t - \lambda_2(y - l)) dl - \int_{y}^{d_2} i_W(l, t + \lambda_2(y - l)) dl \right], \]
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\[ i_{W2}(y, t) = -\frac{1}{2z_02}W_{A2}(t - \tau_2 + \lambda_2y) + \frac{1}{2z_02}W_{B2}(t - \lambda_2y) \]

\[ - \frac{G_{W2}}{2} \left[ \int_0^y v_{W2}(l, t - \lambda_2(y - l)) \, dl - \int_y^{d_2} v_{W2}(l, t + \lambda_2(y - l)) \, dl \right] \]

\[ - \frac{R_{W2}}{2z_02} \left[ \int_0^y i_{W2}(l, t - \lambda_2(y - l)) \, dl + \int_y^{d_2} i_{W2}(l, t + \lambda_2(y - l)) \, dl \right], \]

\[ v_{W1}(0, t) = v_1(t), \quad v_{W1}(d, t) = v_2(t), \]

\[ v_{W2}(0, t) = v_2(t), \quad v_{W2}(d, t) = v_3(t), \]

\[ 0 \leq x \leq d_1, \quad 0 \leq y \leq d_2, \quad 0 \leq t \leq T_e, \]

where \( R_{Wj}, L_{Wj}, C_{Wj}, G_{Wj} \) \((j = 1, 2)\) are respectively the distributed parameters of the first and second lines, and \( \lambda_j = \sqrt{L_{Wj}C_{Wj}} \) \((j = 1, 2)\).

Fig. 3. A circuit with multiple RLCG transmission lines.

Thus, the general form of equations on a circuit system with RLCG transmission lines should be a system of nonlinear IDAEs with multiple constant delays as follows

\[ C(t) \frac{dx(t)}{dt} + G(x(t), t) + DW(t - \tau) \]

\[ + E \int_0^d y(l, t - \lambda l) \, dl + F \int_0^d y(l, t - \tau + \lambda l) \, dl = b(t), \quad (14) \]
Ax(t) + W(t) + BW(t - τ)
+ H \int_0^d y(l, t - λl)dl + L \int_0^d y(l, t - τ + λl)dl = 0, \quad (15)
y(l, t) + PW(t - λl) + QW(t - τ + λl)
+ M \int_0^l y(r, t - λ(l - r)) dr + N \int_0^l y(r, t + λ(l - r)) dr = 0, \quad (16)
y(0, t) = S_1 x(t), \quad y(d, t) = S_2 x(t), \quad y(l, θ) = ψ(l, θ), \quad \text{for} \quad -τ ≤ θ < 0,
x(0) = x_0, \quad W(θ) \equiv ϕ(θ), \quad \text{for} \quad -τ ≤ θ < 0,
0 ≤ l ≤ d, \quad 0 ≤ r ≤ l, \quad t ∈ [0, T_e],

where C(⋅) ∈ R^{n×n} is a matrix-valued function, A, S_1, S_2 ∈ R^{2m×n}, D, E, F ∈ R^{n×2m}, B, H, L, P, Q, M, N ∈ R^{2m×2m}, G(⋅, ⋅) ∈ R^n is a nonlinear function, and for any t and l the functions x(t) ∈ R^n, y(l, t - τ), W(t - τ) ∈ R^{2m} are to be computed in which

\[y(l, t - τ) = [v_1(l_1, t - τ_1), i_1(l_1, t - τ_1), \cdots, v_m(l_m, t - τ_m), i_m(l_m, t - τ_m)]^t,\]
\[W(t - τ) = [W_{A1}(t - τ_1), W_{B1}(t - τ_1), \cdots, W_{Am}(t - τ_m), W_{Bm}(t - τ_m)]^t,\]

where \(l = [l_1, \cdots, l_m]^t\) and \(τ = [τ_1, \cdots, τ_m]^t\). Further, \(λ = [λ_1, \cdots, λ_m]^t > 0\), \(d = [d_1, \cdots, d_m]^t > 0\), \(τ = λd = [τ_1, \cdots, τ_m]^t > 0\), where \(τ_j = λjd_j\) (\(j = 1, \cdots, m\)), \(r = [r_1, \cdots, r_m]^t\), and \(λl = [λ_1l_1, \cdots, λ_ml_m]^t\).

For the above circuit system, b(⋅) ∈ R^n is a known input vector function, \(x_0\) is an initial value, and \(ϕ(θ)\) and \(ψ(l, θ)\) are initial states of the RLCG transmission line system such that

\[ϕ(θ) = [W_{A1}(θ_1), W_{B1}(θ_1), \cdots, W_{Am}(θ_m), W_{Bm}(θ_m)]^t,\]
\[ψ(l, θ) = [v_1(l_1, θ_1), i_1(l_1, θ_1), \cdots, v_m(l_m, θ_m), i_m(l_m, θ_m)]^t,\]

in which \(-τ_j ≤ θ_j < 0\) (\(1 ≤ j ≤ m\)). In practical application the initial values
$x_0, W(0)$ are consistent, that is,

$$Ax_0 + W(0) + BW(−τ) + H \int_0^d y(−λl)dl + L \int_0^d y(−τ + λl)dl = 0.$$ 

Moreover, by invoking the property of transmission lines we should also assume that the form of $B$ in (15) is a block diagonal matrix such that

$$B = \begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix} \in R^{2m \times 2m},$$

where $I_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For the lossless case ($R = G = 0$), the mathematical model and its relaxation solutions in function space are provided in [4].

4 Summary

We have presented a new time-domain model on $RLCG$ transmission lines. The circuit system with distributed elements is described by nonlinear integral-differential-algebraic equations with multiple constant delays. In theory, the new approach directly leads to solution of the circuit system in time-domain and the general-purpose circuit simulators can be then used to solve the system.

References