Alternating-direction method for a mildly nonlinear elliptic equation with nonlocal integral conditions

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Abstract. The present paper deals with a generalization of the alternating-direction implicit (ADI) method for the two-dimensional nonlinear Poisson equation in a rectangular domain with integral boundary condition in one coordinate direction. The analysis of results of computational experiments is presented.

Keywords: elliptic equation, nonlocal integral conditions, finite-difference method, alternating-direction method, convergence of iterative method.

1 Introduction

In this paper, we consider finite-difference approximations for the following nonlinear elliptic equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y, u), \quad (x, y) \in \Omega,$$

with Dirichlet boundary conditions in one coordinate direction

$$u(x, 0) = \mu^l(x), \quad u(x, L_y) = \mu^r(x), \quad x \in [0, L_x],$$

and with nonlocal integral conditions in another coordinate direction:

$$u(0, y) = \gamma_0 \int_0^{L_x} u(x, y) \, dx + \nu^l(y), \quad y \in [0, L_y],$$

$$u(L_x, y) = \gamma_1 \int_0^{L_x} u(x, y) \, dx + \nu^r(y), \quad y \in [0, L_y].$$

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where \( \Omega = (0, L_x) \times (0, L_y) \) is a rectangular domain, \( \gamma_0 \) and \( \gamma_1 \) are given constants.

The function \( f(x, y, u) \) satisfies the condition

\[
\frac{\partial f}{\partial u} \leq 0.
\] (5)

The investigation of methods for solving an equation, commonly referred to as mildly nonlinear, began long ago. One of the first articles where the finite difference method has been studied is [1].

The finite difference scheme of high-order accuracy for the stationary problem with a Dirichlet boundary condition was investigated in [2].

The purpose of this paper is to find numerical solution to this equation with special-type nonlocal conditions. Nonlocal boundary conditions (3)–(4), which can be called nonlocal conditions according to one variable, are that of the typical nonlocal conditions. Currently, they are intensively researched. Theoretical investigation of problems with different types of nonlocal boundary conditions is an actual problem, and recently much attention has been paid to them in the scientific literature.

Elliptic equation with integral conditions of another type that of (3)–(4) is the object of study in the works [3–12]. Various statements of different problems with nonlocal conditions and research methods can be found in [11, 13–27].

The remaining part of this paper is organized as follows. In Section 2, we formulate a difference problem and write the alternating-direction implicit method. In Sections 3 and 4, we present the analysis of convergence and the results of numerical experiments. Section 5 contains some brief conclusions and comments.

## 2 Statement of a difference problem. ADI method

In the domain \( \Omega \) we consider the grids:

\[
\omega^h := \{ x_0 = 0, x_1, \ldots, x_n = L_x \}, \quad h_x = x_i - x_{i-1} = L_x / n, \\
\omega^h_y := \{ y_0 = 0, y_1, \ldots, y_m = L_y \}, \quad h_y = y_j - y_{j-1} = L_y / m, \\
\omega^h_x := \{ x_1, \ldots, x_{n-1} \}, \quad \omega^h_y := \{ y_1, \ldots, y_{m-1} \}.
\]

In the closed domain \( \overline{\Omega} \) we consider the rectangular grids \( \omega^h := \omega^h_x \times \omega^h_y, \omega^h := \omega^h_x \times \omega^h_y \) and \( \partial \omega^h := \overline{\omega^h} \setminus \omega^h \).

If \( \omega \) is one of these grids, we define the space of grid functions \( F(\omega) \).

We introduce second order central difference operators \( \delta_x^2 \) and \( \delta_y^2 \):

\[
\delta_x^2 u_{i,j} := \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2}, \quad \delta_y^2 u_{i,j} := \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2}.
\]

The function \( f \) is approximated by grid function \( f_{ij} \) on the grid \( \omega^h \), functions \( \nu^l, \nu^r \) by \( \nu^l_{ij}, \nu^r_{ij} \) on the grid \( \omega^h_y \) and functions \( \mu^l, \mu^r \) by \( \mu^l_{i}, \mu^r_{i} \) on the grid \( \omega^h_x \).

Equations (1)–(4) are replaced with finite-difference equations:

\[-(\delta_x^2 + \delta_y^2)u_{ij} = f_{ij}(u_{ij}), \quad (x_i, y_j) \in \omega_h^h, \quad (6)\]

\[u_{0j} = \gamma_0 h_x \left( \frac{u_{0j} + u_{nj}}{2} + \sum_{i=1}^{n-1} u_{ij} \right) + \nu_j^l, \quad (7)\]

\[u_{nj} = \gamma_1 h_x \left( \frac{u_{nj} + u_{nj+1}}{2} + \sum_{i=1}^{n-1} u_{ij} \right) + \nu_j^r, \quad (8)\]

\[u_{i0} = \mu_{i1}, \quad u_{im} = \mu_{i1}, \quad x_i \in \omega_h^h. \quad (9)\]

Now we write the Peaceman–Rachford alternating-direction implicit method [28] for the system (6)–(9) as follows:

\[u_{k+1/2}^{i,j} - u_{ij}^k \over \tau_{k+1} = \delta_x^2 u_{ij}^{k+1/2} + \delta_y^2 u_{ij}^k + f_{ij}(u_{ij}^k), \quad i = 1, \ldots, n - 1, \quad (10)\]

\[u_{ij}^{k+1/2} - u_{ij}^k \over \tau_{k+1} = \delta_x^2 u_{ij}^{k+1/2} + \delta_y^2 u_{ij}^{k+1} + f_{ij}(u_{ij}^{k+1/2}), \quad j = 1, \ldots, m - 1, \quad (11)\]

where \(\tau_{k+1}\) are parameters.

For each fixed value \(i = 1, \ldots, n - 1\), we solve equation (11) with boundary conditions

\[u_{i0}^{k+1/2} = \mu_{i1}, \quad u_{im}^{k+1/2} = \mu_{i1}. \quad (12)\]

For each fixed value \(j = 1, \ldots, m - 1\), we solve equation (10) with nonlocal boundary conditions

\[u_{ij}^{k+1} = \gamma_0 h_x \left( \frac{u_{ij}^{k+1} + u_{nj+1}}{2} + \sum_{i=1}^{n-1} u_{ij}^{k+1} \right) + \nu_j^l, \quad (13)\]

\[u_{nj}^{k+1} = \gamma_1 h_x \left( \frac{u_{nj}^{k+1} + u_{nj+1}}{2} + \sum_{i=1}^{n-1} u_{ij}^{k+1} \right) + \nu_j^r. \quad (14)\]

### 3 Analysis of the convergence

We investigate the convergence of the ADI method. Let us write the system of difference equations (6)–(9) in the matrix form. We consider two one-dimensional difference problems with nonlocal or homogeneous Dirichlet conditions

\[v_{i-1} - 2v_i + v_{i+1} \over h_x^2 = p_i, \quad i = 1, \ldots, n - 1, \quad (15)\]

\[v_0 = \gamma_0 h_x \left( \frac{v_0 + v_n}{2} + \sum_{i=1}^{n-1} v_i \right), \quad (16)\]

\[v_n = \gamma_1 h_x \left( \frac{v_0 + v_n}{2} + \sum_{i=1}^{n-1} v_i \right). \quad (17)\]
and

$$w_{j-1} - 2w_j + w_{j+1} = q_j, \quad j = 1, \ldots, m - 1,$$  \hspace{1em} (18)

$$w_0 = 0, \quad w_m = 0,$$  \hspace{1em} (19)

where $p_i, i = 1, \ldots, n - 1$ and $q_j, j = 1, \ldots, m - 1$ are given values.

Let us interpret Eqs. (16)–(17) for each fixed value $j = 1, \ldots, m - 1$ as a system of two equations with the unknown variables $v_0, v_n$. We express these variables in other unknown variables:

$$v_0 = a \sum_{i=1}^{n-1} v_i, \quad v_n = b \sum_{i=1}^{n-1} v_i,$$  \hspace{1em} (20)

$$a = \frac{\gamma_0 h_x}{D}, \quad b = \frac{\gamma_1 h_x}{D}, \quad D = 1 - \frac{(\gamma_0 + \gamma_1)h_x}{2}.$$  \hspace{1em} (21)

If $h_x$ is small enough $h_x < 2/(\gamma_0 + \gamma_1)$, the determinant $D \neq 0$ and $v_0, v_n$ are expressed by formulas (20)–(21) uniquely.

So we can rewrite (15)–(17) in the matrix form

$$A_x v = p,$$  \hspace{1em} (22)

where $A_x$ is the $(n - 1)$ order matrix

$$A_x = \frac{1}{h_x^2} \begin{pmatrix} -2 + a & 1 + a & a & \ldots & a & a \\ 1 & -2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -2 & 1 \\ b & b & b & \ldots & 1 + b & -2 + b \end{pmatrix}.$$  \hspace{1em} (23)

Rewriting the system (18)–(19) in the form

$$A_y w = q,$$  \hspace{1em} (24)

we define $A_y$ as an $(m - 1)$-order tridiagonal matrix

$$A_y = \frac{1}{h_y^2} \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -2 & 1 \\ 0 & 0 & 0 & \ldots & 1 & -2 \end{pmatrix}.$$  \hspace{1em} (25)

Now we can define matrices $A_1, A_2$ and $I$ of order $(n - 1)(m - 1)$ using the Kronecker (tensor) product of matrices:

$$A_1 = -I_{n-1} \otimes A_x, \quad A_2 = -A_y \otimes I_{m-1}, \quad I = I_{m-1} \otimes I_{n-1},$$
where $I_k$ is the identity matrix of order $k$.

Now the iteration method (10)–(11) with boundary conditions (12)–(14) is equivalent to the following iteration method:

\begin{align}
(I + \tau_{k+1} A_1) u^{k+1/2} &= (I - \tau_{k+1} A_2) u^k + \tau_{k+1} f(u^k), \\
(I + \tau_{k+1} A_2) u^{k+1} &= (I - \tau_{k+1} A_1) u^{k+1/2} + \tau_{k+1} f(u^{k+1/2}).
\end{align}

(26)

(27)

Let us define $u^* = \{u^*_{ij}\}$ as the exact solution of system (6)–(9) and

\[ z^k = u^* - u^k. \]

(28)

According to (26)–(27) the following system of equations is true for the error $z^k$

\begin{align}
(I + \tau_{k+1} A_1) z^{k+1/2} &= (I - \tau_{k+1} A_2) z^k - \tau_{k+1} D_1 z^k, \\
(I + \tau_{k+1} A_2) z^{k+1} &= (I - \tau_{k+1} A_1) z^{k+1/2} - \tau_{k+1} D_2 z^{k+1/2},
\end{align}

(29)

(30)

where $D_l, l = 1, 2$ are diagonal matrices with diagonal elements

\[ d_l = \{d^l_{ij}\} = -\partial f(\tilde{u}^l_{ij})/\partial u, \quad l = 1, 2, \]

and $\tilde{u}^l$ is an intermediate point.

Let us indicate the basic properties of matrices $A_x, A_y$ and $A_1, A_2$:

1. $A_y$ is a symmetric matrix. All the eigenvalues of $A_y$ are positive and distinct [29].

   The eigenvalues of the matrix $A_y$ are given by (see, [29]):

   \[ \lambda_j = \frac{4}{h_y^2} \sin^2 \frac{\pi j h_y}{2}, \quad j = 1, \ldots, m - 1. \]

   (31)

2. $A_x$ is a nonsymmetric matrix (it becomes symmetric iff $\gamma_0 = \gamma_1 = 0$, namely, if there are no nonlocal conditions). Its eigenvalues are given in [30].

   • If $\gamma_0 + \gamma_1 = 0$, then there exists one single eigenvalue $\lambda = 0$ and all the other remaining eigenvalues are positive.

   • If $\gamma_0 + \gamma_1 > 2$ and $h_x < 2/(\gamma_0 + \gamma_1)$, then there exists one single eigenvalue $\lambda < 0$

     \[ \lambda = -\frac{4}{h_x^2} \sinh^2 \frac{\beta h_x}{2}, \]

     where $\beta$ is the unique positive root of the equation

     \[ \tanh \frac{\beta}{2} - \frac{2}{h_x (\gamma_0 + \gamma_1)} \tanh \frac{\beta h_x}{2} = 0, \]

     and all the other eigenvalues are positive.
• If \( \gamma_0 + \gamma_1 < 2 \), then all eigenvalues are positive

\[
\lambda_i = \frac{4}{h^2} \sinh^2 \frac{\alpha_i h_x}{2}, \quad i = 1, \ldots, n - 1,
\]

where some of \( \alpha_i \) doesn’t depend on \( \gamma_0 \) and \( \gamma_1 \), i.e.,

\[
\alpha_i = 2i\pi, \quad i = 1, \ldots, \left[ \frac{n - 1}{2} \right],
\]

and the other \( \alpha_i \) are the roots of the equation

\[
\tan \frac{\alpha}{2} - \frac{2}{h_x(\gamma_0 + \gamma_1)} \tan \frac{\alpha h_x}{2} = 0
\]

in the interval \((0, n\pi)\).

3. With all \( \gamma_0, \gamma_1 \) values the matrices \( A_1 \) and \( A_2 \) are commutative [9]

\[
A_1 A_2 = A_2 A_1 = -A_y \otimes A_x.
\]

4. With all \( \gamma_0, \gamma_1 \) values the matrices \( A_x, A_y \) are of simple structure. Therefore the matrices \( A_1, A_2, A_1 + A_2, A_1 A_2, A_2 A_1 \) have the same system of eigenvectors [9].

Let us now write the iteration method (29)–(30) as a matrix equation:

\[
z^{k+1} = Sz^k,
\]

where

\[
S = (I + \tau_{k+1} A_2)^{-1}(I - \tau_{k+1}(A_1 + D_2))(I + \tau_{k+1} A_1)^{-1}(I - \tau_{k+1}(A_2 + D_1)).
\]

**Theorem 1.** If \( \gamma_0 + \gamma_1 < 2 \) and \( \tau_{k+1} > 0 \) are small enough numbers, then the iterative method (10)–(11) is convergent.

**Proof.** In order to prove the convergence of the iterative method (10)–(11), it suffices to prove that \( \|z^k\| \rightarrow 0 \) as \( k \rightarrow \infty \).

Firstly, we consider the case \( f(x, y, u) = -Cu \), where \( C \geq 0 \) is constant. Then \( D_1 \) and \( D_2 \) are diagonal matrices with element \( C \) on the diagonal. So we see that all the four factors in the expression of matrix \( S \) (37) have the same system of eigenvectors. Thus,

\[
\lambda(S) = \frac{(1 - \tau_{k+1}(\lambda(A_1) + C))(1 - \tau_{k+1}(\lambda(A_2) + C))}{(1 + \tau_{k+1}\lambda(A_1))(1 + \tau_{k+1}\lambda(A_2))}.
\]

If \( \gamma_0 + \gamma_1 < 2 \), then \( \lambda(A_1) > 0, \lambda(A_2) > 0 \). Therefore

\[
|\lambda(S)| < 1
\]
with \( \tau_{k+1} > 0 \) sufficiently small, namely,

\[
\tau_{k+1} < \frac{2}{C}. \tag{40}
\]

Now we consider \( f(x, y, u) \neq -Cu \), but \( \partial f/\partial u \leq 0 \). Since the eigenvalues of any matrix are continuous functions of elements of the matrices, the inequalities \( \lambda(A_1) > 0 \), \( \lambda(A_2) > 0 \), \( |\lambda(S)| < 1 \) are true for \( D_1 = D_2 = 0 \), hence there exists such a number \( \tau_0 > 0 \) that inequality (39) is true for all \( \tau_{k+1} \in (0, \tau_s] \). The theorem is proved.

In practice, it is important to know what value \( \tau_0 \) takes and how fast the iterative method (10)–(11) converges.

These questions are still uninvestigated theoretically. In the next section, we partially answer these questions using computer simulation methods.

Let us denote the smallest and the largest eigenvalues of the matrices \( A_1, A_2 \) by \( \delta_1, \Delta_1, \delta_2, \Delta_2 \). From the expressions of the eigenvalues of \( A_1, A_2 \) we obtain

\[
\delta_1 = \frac{4}{h_x^2} \min_k \sin^2 \alpha_k h_x^2, \quad \Delta_1 = \frac{4}{h_x^2} \max_k \sin \alpha_k h_x^2, \quad \delta_2 = \frac{4}{h_y^2} \sin^2 \pi h_y^2, \quad \Delta_2 = \frac{4}{h_y^2} \cos^2 \pi h_y^2.
\]

4 Numerical experiment

We consider a model problem (1)–(4) [31] in a unit square domain \([0, 1] \times [0, 1]\).

The right-hand side (RHS) function \( f(x, y, u) \) is given by

\[
f(x, y, u) = \frac{\pi^2}{4} u(1 - u) + g(x, y), \tag{41}
\]

where

\[
g(x, y) = 2 \sin \left( \frac{\pi}{2} y \right) + \frac{\pi^2}{4} (1 - x^2) \sin^2 \left( \frac{\pi}{2} y \right). \tag{42}
\]

The exact solution to this test problem is given by

\[
u(x, y) = (1 - x^2) \sin \left( \frac{\pi}{2} y \right). \tag{43}
\]

The initial and boundary conditions were prescribed to satisfy the exact solution (43).

We consider uniform grids with different mesh sizes \( h = h_x = h_y \) and analyze the convergence and accuracy of the computed solution from the present ADI scheme. We compute the maximum norm of the error of the numerical solution with respect to the exact solution, which is defined as

\[
\varepsilon_h = \max_{j=1, \ldots, m} \max_{i=1, \ldots, n} \left| u(x_i, y_j) - u_{ij} \right|.
\]

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We define the number $p$ as

$$p = \frac{\varepsilon h}{\varepsilon^2 h},$$

which theoretically must be approximately $p \approx 4$.

The results of the numerical test are listed in Table 1. Note that inequality (5) in the neighborhood of the point $x = 0$, $y = 1$ is not satisfied.

Table 1. The errors for different $\gamma_0$, $\gamma_1$ in the case of the RHS function (41).

<table>
<thead>
<tr>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$h$</th>
<th>$\varepsilon_h$</th>
<th>$p$</th>
<th>number of iter.</th>
</tr>
</thead>
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<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.25</td>
<td>$1.08749 \cdot 10^{-4}$</td>
<td>3.7429</td>
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<tr>
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<td>1.85111 \cdot 10^{-5}</td>
<td>3.9736</td>
<td>24</td>
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<td></td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$h$</th>
<th>$\varepsilon_h$</th>
<th>$p$</th>
<th>number of iter.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-1.0</td>
<td>0.25</td>
<td>$1.23522 \cdot 10^{-4}$</td>
<td>3.8152</td>
<td>15</td>
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<td>0.0625</td>
<td>8.25150 \cdot 10^{-5}</td>
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<td>2.07282 \cdot 10^{-5}</td>
<td>3.9808</td>
<td>24</td>
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<table>
<thead>
<tr>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$h$</th>
<th>$\varepsilon_h$</th>
<th>$p$</th>
<th>number of iter.</th>
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<td>1.0</td>
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<td>4.0392</td>
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<tr>
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<td>4.95807 \cdot 10^{-4}</td>
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<tr>
<td>0.0625</td>
<td>1.25950 \cdot 10^{-4}</td>
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<thead>
<tr>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$h$</th>
<th>$\varepsilon_h$</th>
<th>$p$</th>
<th>number of iter.</th>
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<td>4.35843 \cdot 10^{-4}</td>
<td>0.9973</td>
<td>24</td>
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</table>

In the second test problem, we choose $f(x, y, u)$ as

$$f(x, y, u) = -Cu + \sin \left( \frac{\pi}{2} y \right) \left( 2 + \left( 1 - x^2 \right) \frac{\pi^2}{4} \right) + C \left( 1 - x^2 \right) \sin \left( \frac{\pi}{2} y \right). \tag{44}$$

The exact solution to this test problem is given by (43).

Table 2 presents the performance of the algorithm for various values of constant $C$. Note that for large values of $|\gamma_0|$, $|\gamma_1|$ the error increases. The function (41) holds the condition (5) only in the part of the domain.

Table 2. The errors for different $\gamma_0$, $\gamma_1$ and $C$ in the case of the RHS function (44).

<table>
<thead>
<tr>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$h$</th>
<th>$\varepsilon_h$</th>
<th>$p$</th>
<th>number of iter.</th>
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<td>0.0</td>
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<td>0.125</td>
<td>2.84430 \cdot 10^{-4}</td>
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<td>0.0625</td>
<td>7.15912 \cdot 10^{-5}</td>
<td>3.9740</td>
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<td>0.03125</td>
<td>1.80151 \cdot 10^{-5}</td>
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</tbody>
</table>
In both cases the set of optimal iterative parameters of the ADI method was chosen according to the monograph [29] where symmetric matrices of an iterative process are used.

5 Conclusions and remarks

The ADI method can be used for a mildly nonlinear Poisson equation. Nonlocal integral conditions with $\gamma_0 + \gamma_1 < 2$ never cause more problems than the classical conditions both in the number of iterations and precision of the solution. But these conditions affect the region of convergence of the method. The convergence domain depends essentially on the coefficients of nonlocality. The values of parameters $\gamma_0$ and $\gamma_1$ in nonlocal boundary conditions are essential for the stability of the ADI method. The results of the numerical experiment are in good agreement with the existing theoretical results for a two-dimensional Poisson equation in a rectangle domain with an integral boundary condition in one coordinate direction [10].

References


